Econ 802

Answers to Midterm 1

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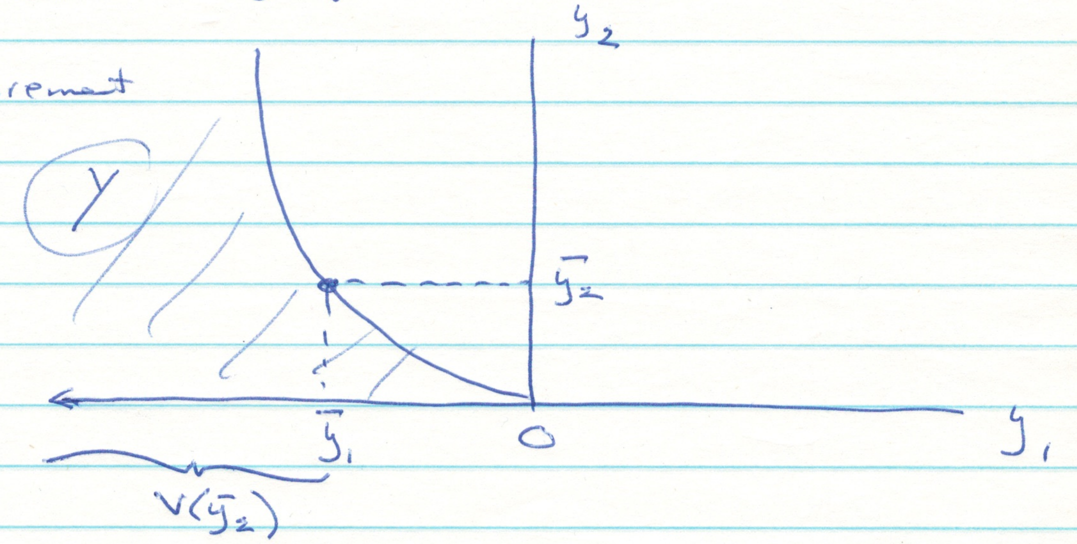
1. (a) The statement is incorrect. Consider a production possibilities set Y where $y = (y_1, y_2) \in Y$ iff $y_1 \leq 0$ and $0 \leq y_2 \leq y_1^2$. The graph is

The input requirement set $V(\bar{y}_2)$ is the set of all input levels $y_1 \leq \bar{y}_1$ (with a sign change to make the input quantities non-negative).



All such input requirement sets are convex. However, you can choose any pair of points along the upper boundary of Y and the line segment between them will be above Y, so this set is not convex.

(b) We can't be certain of this. It is true that if the production function is differentiable and the sufficient second order conditions hold, we can show that the substitution matrix $\frac{\partial x}{\partial w}$ for the unconditional input demands is negative definite so all of the diagonal terms are strictly negative and the input demand curves are downward sloping. Similar assumptions would allow us to show that the output supply curve is upward sloping. However, if the sufficient SOC do not hold, we can't use the implicit function theorem to manipulate the FOC. In this case we have to get comparative static results using the algebraic method or Hotelling's Lemma. Neither of these methods rules out horizontal portions of the input demand or output supply curves.

1(c) Half credit for knowing that $P = MC$ is a necessary condition for profit max. Full credit for also knowing that the competitive firm is a price taker and does not choose p. Instead it takes p as given and adjusts output until $MC = P$.

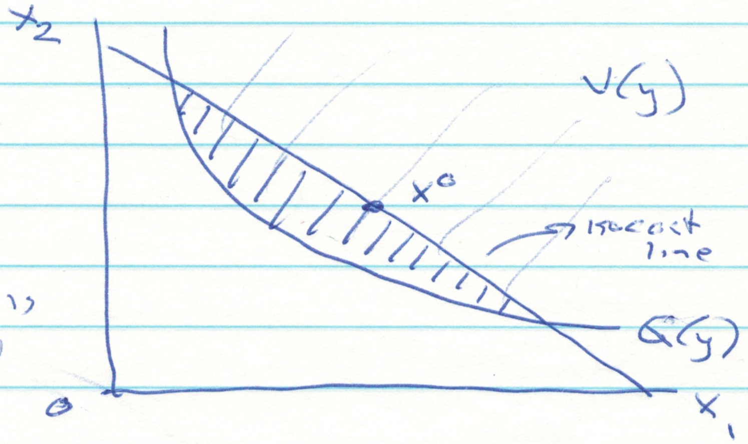
2. (a) No, you cannot be confident that a solution exists. If $\alpha + \beta = 1$ then we have CRS. We showed in class that if it is possible to have positive profit at some x^0, then CRS implies that profit is unbounded:

$$pf(tx^0) - w(tx_0) = t\,[\underbrace{pf(x^0) - wx^0}_{>0}]$$

Profit has no upper bound because we can choose any $t > 0$. Similar problems arise when $\alpha + \beta > 1$ so we have IRS.

However, it is true that when $\alpha + \beta < 1$ (DRS) then the FOC for profit max have a unique solution and it is also true that the sufficient SOC holds globally. This is enough to guarantee that a solution exists and is unique.

(b) Yes, you can be confident that a solution exists. Suppose you want to minimize the cost of producing some output y^0. There will always be some x^0 that is large enough to to produce y^0 (The set $V(y^0)$ is non-empty). Let The input expenditure from x^0 be wx^0. Now consider The set of all x vectors such That $wx \le wx^0$ and $x \in V(y^0)$. The graph is:



We know any solution must be in The heavy shaded area (including The boundaries). This is a compact set (it is closed and bounded), and we are minimizing a continuous function wx on This set. Thus we can use The Weierstrasse Theorem to prove That There is some x^* in The set at which wx reaches a minimum.

For The Cobb-Douglas case, we can be sure The solution is unique because The input requirement set is strictly convex (you can prove This by studying The curvature of The isoquant mathematically; it bends The right way regardless of The sum $\alpha + \beta$). Alternatively you can show That any point x^* satisfying The FOC (There is only one such point) also satisfies The sufficient SOC for cost min or you can show That The Cobb-Douglas function is strictly quasi-concave.

(c) The Lagrangean is $L = wx - \lambda[f(x) - y]$

FOC: $w_1 = \lambda f_1(x) = \lambda \alpha x_1^{\alpha-1} x_2^{1-\alpha}$ (note: $\beta = 1-\alpha$

$w_2 = \lambda f_2(x) = \lambda(1-\alpha) x_1^{\alpha} x_2^{-\alpha}$ due to CRS)

Divide top equation by bottom equation to get

$$\frac{w_1}{w_2} = \left(\frac{\alpha}{1-\alpha}\right)\left(\frac{x_2}{x_1}\right) \Rightarrow \frac{w_1 x_1}{w_2 x_2} = \frac{\alpha}{1-\alpha}$$

The cost share for input 1 is $\frac{w_1 x_1}{w_1 x_1 + w_2 x_2}$

$$= \frac{\frac{w_1 x_1}{w_2 x_2}}{\frac{w_1 x_1}{w_2 x_2} + 1}$$

$$= \frac{\frac{\alpha}{1-\alpha}}{\frac{\alpha}{1-\alpha} + 1} = \frac{\frac{\alpha}{1-\alpha}}{\frac{1}{1-\alpha}} = \alpha.$$

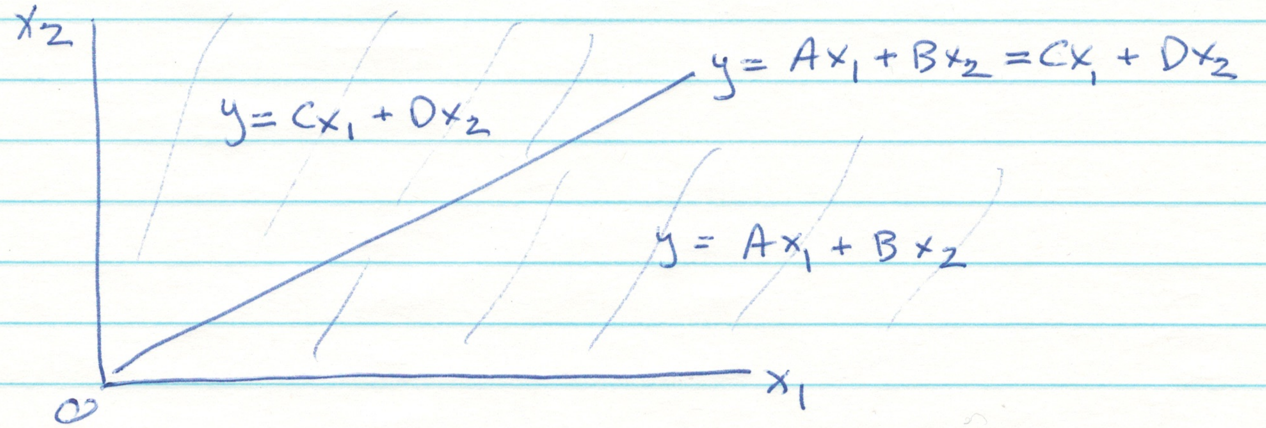
The cost share for input 2 is $1-\alpha$ (prove in a similar way). Economic interpretation: for the Cobb-Douglas function, the cost shares are equal to the exponents (in the CRS case) and thus they are fixed. They don't depend on prices, quantities, or output. This is related to the fact that the Cobb-Douglas is a special case of the CES function where the elasticity of substitution is $\sigma = 1$ (neither elastic nor inelastic; the price and quantity effects cancel out when we compute cost shares).

3(a) Output is $Ax_1 + Bx_2$ whenever $Ax_1 + Bx_2 \geq Cx_1 + Dx_2$

$$\Rightarrow x_2(B-D) \geq x_1(C-A)$$
$$\Rightarrow x_2(D-B) \leq x_1(A-C)$$
$$\Rightarrow x_2 \leq \frac{x_1(A-C)}{(D-B)}$$

due to $A > C$, $D > B$

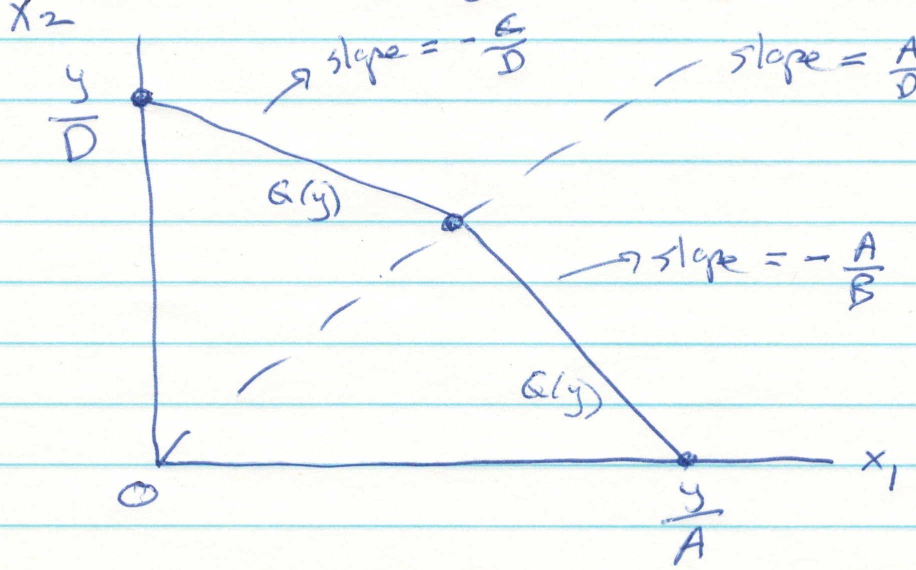
The ray from the origin has slope $\frac{A-C}{D-B} > 0$



Anywhere on or below the ray we get $y = Ax_1 + Bx_2$
" " " above " " we get $y = Cx_1 + Dx_2$

(b) Start from some point along the ray $x_2 = \frac{x_1(A-C)}{D-B}$
Let the resulting output be y. Above the ray we have $y = Cx_1 + Dx_2 \Rightarrow x_2 = \frac{y - Cx_1}{D}$
Below the ray $y = Ax_1 + Bx_2 \Rightarrow x_2 = \frac{y - Ax_1}{B}$

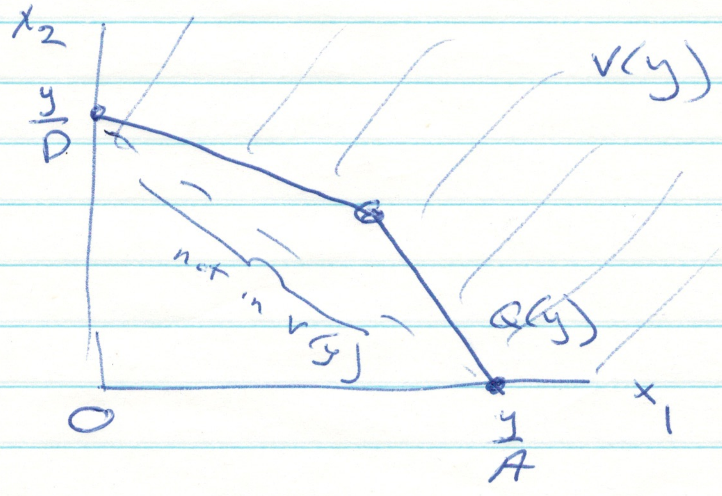
y = constant.



where the upper segment is flatter because $\frac{C}{D} < \frac{A}{B}$

due to $A > C$, $D > B$

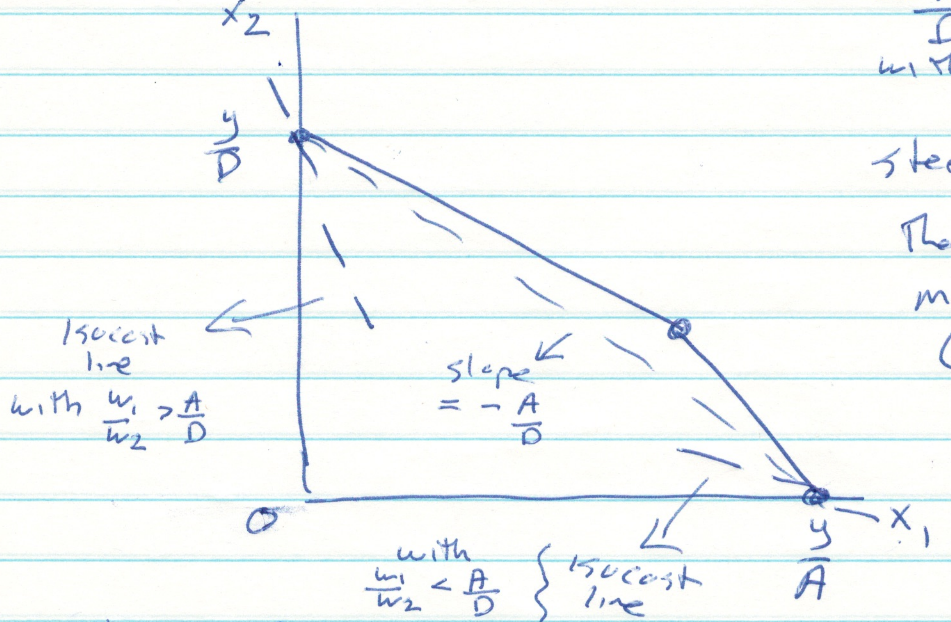
The set $V(y)$ is not convex: you can choose the intercepts of the isoquant $Q(y)$, which are both in $V(y)$. The points between them are not.



Because $V(y)$ is not convex, it is not strictly convex.

(c) For a fixed y, consider the dashed line connecting the intercepts $\left(\frac{y}{A}, 0\right)$ and $\left(0, \frac{y}{D}\right)$. This line has the slope $-\frac{A}{D}$. If the isocost line with the slope $-\frac{w_1}{w_2}$ is steeper than this (so $\frac{w_1}{w_2} > \frac{A}{D}$) then the unique cost-minimizing point is $\left(0, \frac{y}{D}\right)$ which is a boundary solution.

If the isocost line is flatter than $-\frac{A}{D}$, (so $\frac{w_1}{w_2} < \frac{A}{D}$) then the unique cost-minimizing point is $\left(\frac{y}{A}, 0\right)$, which is a boundary solution.



If $\frac{w_1}{w_2} = \frac{A}{D}$ so the isocost line has the same slope as the line segment between the intercepts then there are exactly two solutions, which are the intercepts (again, no interior solution).

4(a) Use Hotelling: $x_1(p, w_1, w_2, x_2) = -\frac{\partial \pi}{\partial w_1}$

$$= -x_2\left[(1-\alpha)\, p^{\frac{1}{1-\alpha}}\, \alpha^{\frac{\alpha}{1-\alpha}} \left[-\frac{\alpha}{1-\alpha}\right] w_1^{-\frac{\alpha}{1-\alpha}-1}\right]$$

$$= x_2\left(\frac{\alpha p}{w_1}\right)^{\frac{1}{1-\alpha}}$$

Yes, the comparative statics make sense.
If $w_1 \uparrow$ then $x_1 \downarrow$ so the demand curve for input 1 is downward sloping.
If $p \uparrow$ then $x_1 \uparrow$ which makes sense because with x_2 fixed, the only way to have $y \uparrow$ is $x_1 \uparrow$.
(output supply is upward sloping as long as the marginal product of x_1 is positive)

(b) Use Hotelling again: $y(p, w_1, w_2, x_2) = \frac{\partial \pi}{\partial p}$

$$= x_2\left[(1-\alpha)\left(\frac{1}{1-\alpha}\right) p^{\frac{1}{1-\alpha}-1}\left(\frac{\alpha}{w_1}\right)^{\frac{\alpha}{1-\alpha}}\right]$$

$$= x_2\left(\frac{\alpha p}{w_1}\right)^{\frac{\alpha}{1-\alpha}}$$

Again the comparative statics make sense.
If $p \uparrow$ then $y \uparrow$ so output supply is upward sloping.
If $w_1 \uparrow$ then $y \downarrow$. We would expect that $w_1 \uparrow$ implies $x_1 \downarrow$ and since x_2 is fixed, this should reduce output.

4(c) Consider the sign of the coefficient of x_2:

$$(1-\alpha)p^{\frac{1}{1-\alpha}}\left(\frac{\alpha}{w_1}\right)^{\frac{\alpha}{1-\alpha}} - w_2.$$

Recall that $\pi(p, w_1, w_2, x_2)$ is the max profit for a given x_2.

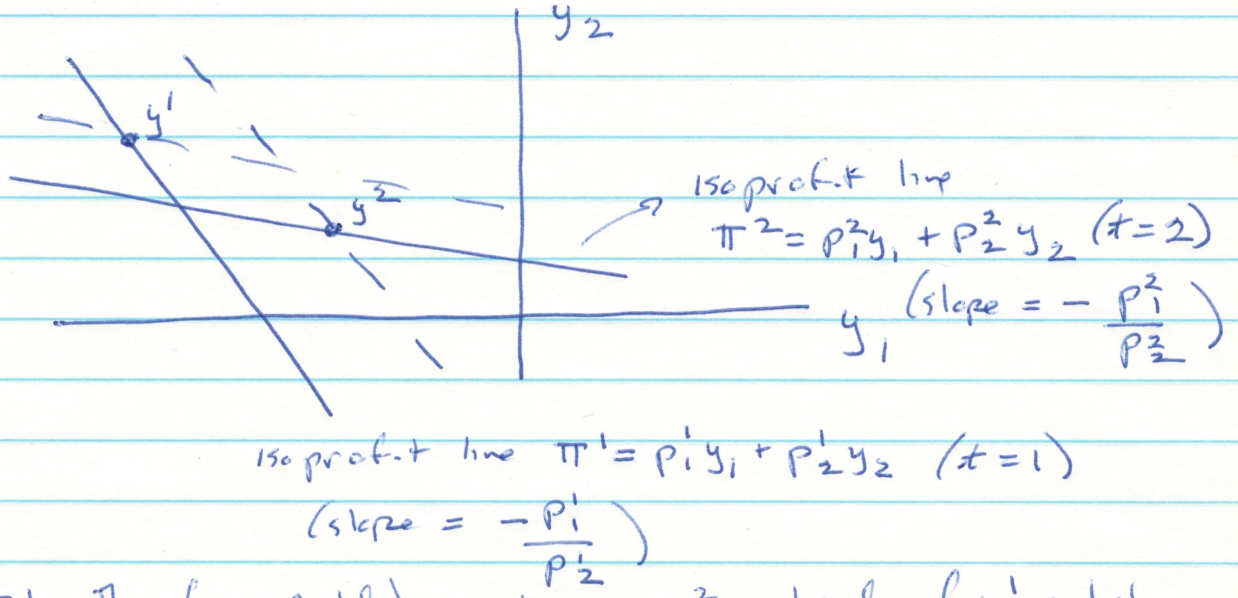
If the coefficient is positive, then we can make profit unboundedly positive in the long run by choosing large x_2 so the long run profit function will not be well-defined.

If the coefficient is zero, then for any level of x_2 the maximum profit will be zero, so in the long run $\pi(p, w) = 0$ which is well defined (although x_2 is indeterminate)

If the coefficient is negative, then max profit for any $x_2 > 0$ is negative; but it is possible to get zero profit by using $x_2 = 0$, so the max profit is again zero and $\pi(p, w) = 0$, which is well-defined (and x_2 is uniquely zero).

[Note: This example was constructed using a Cobb-Douglas production function with CRS, although you didn't need to know that in order to answer the question.]

5(a)



In period $t=1$, the firm could have chosen y^2 instead of y', which would have put it on the higher (dashed) isoprofit line through y^2. This gives $p'y^2 > p'y'$ so y' was not profit maximizing in period 1.

Similarly, in period $t = 2$, the firm could have chosen y^1 instead of y^2, which would have put it on the higher (dashed) isoprofit line through y^1. This gives $p^2y^1 > p^2y^2$ so y^2 was not profit maximizing in period 2.
These observations violate WAPM.

5(b) We know y^* is profit $\underbrace{\text{maximizing at the prices}}_{\text{This follows from } py^* \geq py \text{ for all } y \in Y.}$ p so
$\pi(p) = py^*$

For any $t > 0$, it must also be true that $tpy^* > tpy$ for all $y \in Y$. Or to put this another way: choose any $y' \in Y$ where $y' \neq y^*$, we know $py^* \geq py'$ so it must also be true that $tpy^* \geq tpy'$.
Therefore when the prices are tp we have $(tp)y^* \geq (tp)y$ for all $y \in Y$ and y^* is still optimal.
Thus $\pi(tp) = (tp)y^* = t(py^*) = t\pi(p)$.
This makes sense because multiplying the objective function by a positive constant does not affect the solution to the max problem. Another way to think about it is that the units of currency in which we measure prices are irrelevant for the firm's behavior. So if we multiply all the prices by $t > 0$, we also multiply max profit by t.

(c) From the profit max problem we have $y^* = f(x^*)$ and

FOC: $p\dfrac{\partial f(x^*)}{\partial x} = w$

SOC (suff): $h' \dfrac{\partial^2 f(x^*)}{\partial x^2} h < 0$ for all $h \neq 0$.

In the cost min problem, we have

FOC: $w = \lambda \dfrac{\partial f(x^*)}{\partial x}$ and $f(x^*) = y^*$.

These FOC are satisfied at x^* because we can use the multiplier $p = \lambda$ and we already know that $f(x^*) = y^*$.

The only remaining thing to check is the sufficient SOC for cost min. This is

$$h' \frac{\partial^2 f(x^*)}{\partial x^2} h < 0 \text{ for all } h \neq 0 \text{ such that } \frac{\partial f(x^*)}{\partial x} h = 0.$$

From the suff. SOC for profit max, we know the strict inequality holds for all $h \neq 0$, so this must also be true for the subset of h vectors with $\frac{\partial f(x^*)}{\partial x} \cdot h = 0$. Therefore the sufficient SOC for cost min is satisfied at x^*.

So the statement is true. The economic intuition is that if a firm is maximizing profit and it is producing the output y^*, then it must also be minimizing the cost of producing y^*. If it were not minimizing cost, it could find a cheaper way to produce y^* and increase its profit.

[Note: you didn't need to know this to answer the question, but later we will show that the multiplier λ in the cost min problem is marginal cost, so when we write $p = \lambda$ we are just saying $p = MC$.]